Chapter 6: Residues and Poles

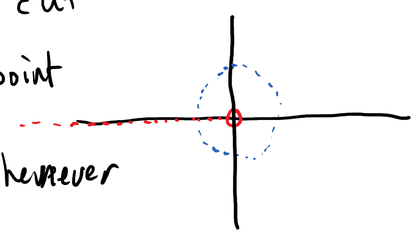
Definition (Isolated Singularity) A singular point z_0 of a function f is *isolated* if there exists a deleted disk $D_\varepsilon(z_0)\setminus\{z_0\}$ on which f is analytic.

Examples

(1) A rational function $F(z) = \dfrac{P(z)}{Q(z)}$ has only isolated singularities. They are the zeros of $Q(z)$.

(2) The principal branch of the logarithm has a singularity at 0. It is not isolated since any deleted disk about 0 contains points on the branch cut

(3) $f(z) = \dfrac{1}{\sin(\frac{\pi}{z})}$ has a singular point



at $z=0$. Also, it has singular points whenever

$$\sin\frac{\pi}{z} = 0 \iff \frac{\pi}{z} = k\pi \quad \text{for } k\in\mathbb{Z}$$

$$\iff z = \frac{1}{k} \quad \cancel{\text{fo}} \quad k\in\mathbb{Z}.$$

The singular point $z=0$ is not isolated. Let $D_\varepsilon(0)\setminus\{0\}$ be a deleted neighborhood about 0. Since $\varepsilon > 0$, choose $k\in\mathbb{N}$ such that $\frac{1}{k} < \varepsilon$. Then $z=\frac{1}{k}\in D_\varepsilon(0)\setminus\{0\}$, but f is not analytic at $z=\frac{1}{k}$.

The singularities $z=\frac{1}{k}$ are isolated since f is analytic on the deleted disk $D_{\frac{1}{k(k+1)}}\left(\frac{1}{k}\right)\setminus\left\{\frac{1}{k}\right\}$. //

$k(k+1)$ //

$\boxed{\text{Definition (Isolated Singularity at } \infty)}$ A function $f(z)$ has an isolated singularity at ∞ if there exists $R>0$ such that f is analytic on the annulus $R<|z|<\infty$.

$\boxed{\text{Definition (Residues)}}$ Let z_0 be an isolated singularity of f so that f is analytic on an annulus

$$\begin{cases} 0<|z-z_0|<R, & z_0 \neq \infty \\ R<|z|<\infty, & z_0=\infty. \end{cases}$$

When $z_0 \neq \infty$, the residue of f at z_0 is the coefficient

$$\operatorname*{Res}_{z=z_0} f(z) \overset{\text{def}}{=} b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$$

in the Laurent series expansion of f. When $z_0=\infty$, the residue of f at ∞ is defined via

$$\operatorname*{Res}_{z=\infty} f(z) \overset{\text{def}}{=} \frac{1}{2\pi i}\int_{C_{R_0}} f(z)\,dz$$

where C_{R_0} is a negatively oriented circle centered at 0 with radius $R_0>R$. //

$\boxed{\text{Example}}$

(1) Compute $\int_C \frac{e^z-1}{z^4}\,dz$ where C is the unit circle w/ positive orientation. Since zero is an isolated singularity of $\frac{e^z-1}{z^4}$, and C is a contour about 0, we need

only compute $\operatorname{Res}_{z=0} \frac{e^z-1}{z^4}$. The function has a Laurent series on $0<|z|<\infty$. We have

$$\frac{e^z-1}{z^4} = \frac{1}{z^4}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1\right) = \frac{1}{z^4}\sum_{n=1}^{\infty} \frac{z^n}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{z^{n-4}}{n!}$$

$$= \frac{1}{z^3} + \frac{1}{2z^2} + \frac{1}{6z} + \sum_{n=0}^{\infty} \frac{z^n}{(n+4)!}$$

Hence, $\operatorname{Res}_{z=0} \frac{e^z-1}{z^4} = \frac{1}{6}$ and $\int_C \frac{e^z-1}{z^4}\,dz = 2\pi i \operatorname{Res}_{z=0} \frac{e^z-1}{z^4}$

$$= \frac{\pi i}{3}.$$

(2) Compute $\int_C \cosh\left(\frac{1}{z^2}\right) dz$ where C is the unit circle w/ positive orientation. The function $\cosh \frac{1}{z^2}$ has an isolated singularity at 0 and it is analytic on the annulus $0<|z|<\infty$. We have

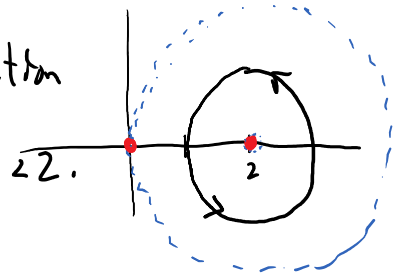
$$\cosh \frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z^2}\right)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{4n}(2n)!} = 1 + \frac{1}{2z^4} + \frac{1}{24z^8} + \cdots +$$

So $\operatorname{Res}_{z=0} \cosh \frac{1}{z^2} = 0$ and $\int_C \cosh \frac{1}{z^2} = 2\pi i \cdot 0 = 0.$

(3) Compute $\int_C \frac{1}{z(z-2)^5}\,dz$ where C is the circle $|z-2|=1$ w/ positive orientation.

We need to compute $\displaystyle\operatorname*{Res}_{z=2} \frac{1}{z(z-2)^5}$. The function has a Laurent series on the annulus $0<|z-2|<2$. Note that this condition implies $\left|\frac{z-2}{2}\right|<1$.



We have

$$\frac{1}{z(z-2)^5} = \frac{1}{(z-2)^5}\left(\frac{1}{2+(z-2)}\right)$$

$$= \frac{1}{2(z-2)^5}\left(\frac{1}{1+\frac{z-2}{2}}\right)$$

$$= \frac{1}{2(z-2)^5}\sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^n}{2^n} = \sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^{n-5}}{2^{n+1}}$$

The residue occurs when $n=4$. Hence, $\displaystyle\operatorname*{Res}_{z=2}\frac{1}{z(z-2)^5} = \frac{1}{32}$.

Hence,

$$\int_C \frac{1}{z(z-2)^5}\,dz = \frac{\pi i}{16}.$$

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Theorem (Residue Theorem) Let C be a positively oriented simple closed contour. If f is analytic everywhere on and interior to C, except at a finite number of singularities $z_1,\dots,z_n$ lying interior to C, then

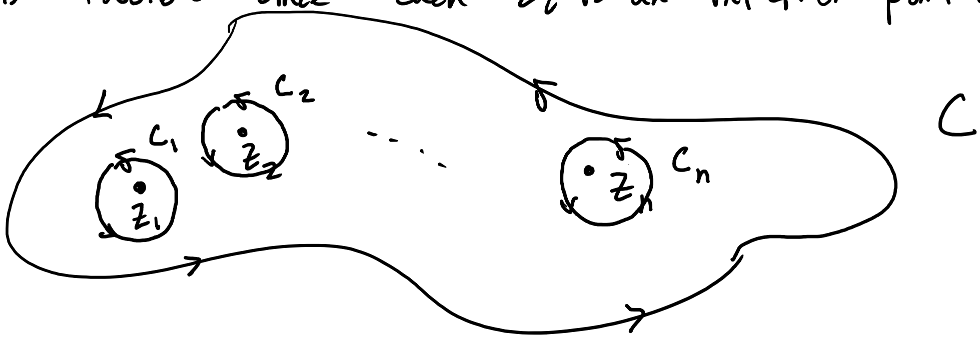
$$\int_C f(z)\,dz = 2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z).$$

Proof. (C.f. Midterm M4) The singularities $z_1, \dots, z_n$ are isolated since there are finitely many. For each $i = 1, \dots, n$, let C_i be a positively oriented circle centered at z_i such that

(1) the regions R_i enclosed by C_i are pairwise disjoint.

(2) the regions R_i enclosed by C_i lies interior to C.

Note that (2) is possible since each z_i is an interior point of C.



Then f is analytic everywhere on $C, C_1, \dots, C_n$ and at all points that are interior to C but exterior to each C_i. By the Generalized Cauchy Goursat theorem,

$$\int_C f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz$$

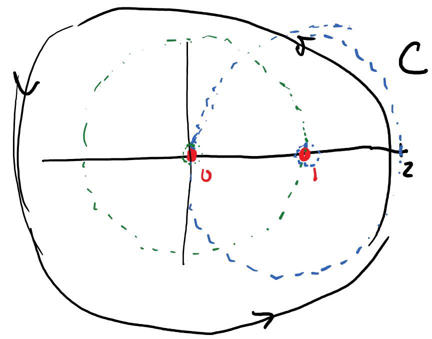
$$= \sum_{i=1}^{n} 2\pi i \operatorname*{Res}_{z=z_i} f(z)$$

$$= 2\pi i \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} f(z)$$

$\blacksquare$

$\boxed{\text{Example}}$ Compute $\displaystyle\int_C \frac{4z-5}{z(z-1)}\,dz$ over the circle $|z| = 2$ with positive orientation. The function has isolated singularities at $z = 0, 1$, both of which lies interior to C. So, we apply the residue theorem. The function has a Laurent series on the annulus $0 < |z| < 1$.

We have



$$\frac{4z-5}{z(z-1)} = \frac{4z-5}{z}\left(\frac{1}{z-1}\right)$$

$$= \frac{5-4z}{z}\left(\frac{1}{1-z}\right)$$

$$= \left(\frac{5}{z}-4\right)\sum_{n=0}^{\infty} z^n = \frac{5}{z}\sum_{n=0}^{\infty} z^n - 4\sum_{n=0}^{\infty} z^n$$

$$= 5\sum_{n=0}^{\infty} z^{n-1} - 4\sum_{n=0}^{\infty} z^n$$

$$= \frac{5}{z} + \sum_{n=0}^{\infty} z^n$$

So $\operatorname{Res}_{z=0} f(z) = 5.$

To compute $\operatorname{Res}_{z=1} f(z)$, we seek a Laurent series on the annulus $0<|z-1|<1$. We have

$$\frac{4z-5}{z(z-1)} = \frac{4z-5}{z-1}\left(\frac{1}{1+(z-1)}\right)$$

$$= \frac{4(z-1)-1}{z-1}\sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

$$= \left(4-\frac{1}{z-1}\right)\sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

$$= 4\sum_{n=0}^{\infty}(-1)^n(z-1)^n - \sum_{n=0}^{\infty}(-1)^n(z-1)^{n-1}$$

$$= 4\sum_{n=0}^{\infty}(-1)^n(z-1)^n - \sum_{n=1}^{\infty}(-1)^n(z-1)^{n-1} - \frac{1}{z-1}$$

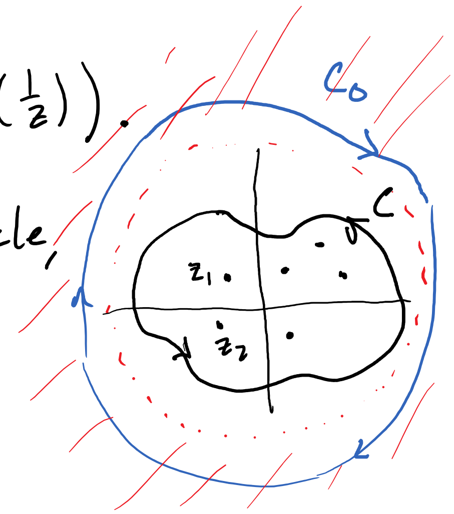
Hence, $\operatorname{Res}_{z=1} f(z) = -1$ and
$$\int_C \frac{4z-5}{z(z-1)}\,dz = 2\pi i(5-1) = 8\pi i. \quad //$$

$\boxed{\text{Theorem (Residues at } \infty)}$ If a function is analytic everywhere on $\mathbb{C}$ except at a finite number of singularities lying interior to a simple closed pos. oriented contour C, then

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right).$$

That is,

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right).$$



Proof. Let C_0 be a negatively oriented circle, centered at 0, whose interior contains C. By Principle of Deformation of Paths,

$$\int_C f(z)\,dz = \int_{-C_0} f(z)\,dz$$

$$= -\int_{C_0} f(z)\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z).$$

To compute $\operatorname*{Res}_{z=\infty} f(z)$, we find the Laurent series of f on an annulus $R < |z| < \infty$ where $\max_{w\in C}|w| < R <$ radius C_0. We have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \frac{1}{z^n}$$

where

$$a_n = \frac{1}{2\pi i}\int_{-C_0} \frac{f(z)}{z^{n+1}}\,dz, \quad b_n = \frac{1}{2\pi i}\int_{-C_0} \frac{f(z)}{z^{-n+1}}\,dz.$$

Then

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n \frac{1}{z^{n+2}} + \sum_{n=1}^{\infty} b_n z^{n-2}, \quad \left(0 < \left|\frac{1}{z}\right| < \frac{1}{R}\right).$$

Note that

$$\operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = b_1 = \frac{1}{2\pi i} \int_{-C_0} f(z)\,dz$$

$$= -\operatorname*{Res}_{z=\infty} f(z).$$

So this proves the claim.

∎

$\boxed{\text{Example}}$ Let C be the circle $|z|=2$ w/ positive orientation. We can compute

$$\int_C \frac{4z-5}{z(z-1)}\,dz$$

using a single residue. Let $f(z) = \frac{4z-5}{z(z-1)}$. This function has singularities at $z=0,1$, both of which lie interior to C. We have

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\left(\frac{\frac{4}{z}-5}{\frac{1}{z}\left(\frac{1}{z}-1\right)}\right) = \frac{1}{z^2}\left(\frac{\frac{4-5z}{z}}{\frac{1}{z}\left(\frac{1-z}{z}\right)}\right)$$

$$= \cancel{\frac{1}{z^2}}\left(\frac{(4-5z)\cancel{z^2}}{z(1-z)}\right)$$

$$= \frac{4-5z}{z(1-z)}.$$

Notice that $\frac{4-5z}{1-z}$ is analytic at zero so it has a Maclaurin series:

$$(4-5z)\frac{1}{1-z} = (4-5z)\sum_{n=0}^{\infty} z^n = 4 + \text{stuff}$$

Hence, the coefficient of $\frac{1}{z}$ in $\frac{1}{z}\cdot\frac{4-5z}{1-z}$ is 4.

So $\operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 4$ and

$$\int_C \frac{4z-5}{z(1-z)}\,dz = 2\pi i \cdot 4 = 8\pi i$$

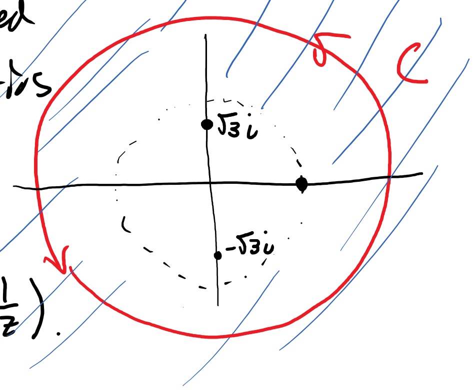
by the theorem. //

$\boxed{\text{Example}}$ The function

Compare w/ Pset 7

$$f(z) = \frac{z^2}{(z-2)(z^2+3)}$$

has no antiderivative on the domain $D = \{ z \in \mathbb{C} : |z| > 2 \}$. Let C be the circle of radius 3 centered at zero. Notice that all three singularities of f lie interior to C. Hence, by the theorem,



$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right).$$

We have

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\left(\frac{\frac{1}{z^2}}{\left(\frac{1-2z}{z}\right)\left(\frac{1+3z^2}{z^2}\right)} \right) = \frac{1}{z^2}\left(\frac{z^3}{z^2(1-2z)(1+3z^2)} \right)$$

$$= \frac{1}{z(1-2z)(1+3z^2)}.$$

The functions $\frac{1}{1-2z}$ and $\frac{1}{1+3z^2}$ are analytic at $z=0$, hence

$$\frac{1}{1-2z} = \sum_{n=0}^{\infty} (2z)^n \quad \text{and} \quad \frac{1}{1+3z^2} = \sum_{n=0}^{\infty} (-1)^n (3z^2)^n$$

and so the Taylor series of the product $\left(\frac{1}{1-2z}\right)\left(\frac{1}{1+3z^2}\right)$ has constant term equal to 1 and hence

$$\operatorname*{Res}_{z=0} \frac{1}{z(1-2z)(1+3z^2)} = 1.$$

Hence, $\int_C f(z)\,dz = 2\pi i$ and by the fund. thm. of contour integrals, f has no antiderivative on D. //

Classifying Isolated Singularities

Recall: If f has an isolated singularity at $z_0 \in \mathbb{C}$, then f has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n}$$

on some annulus $0 < |z-z_0| < R$. The sum

$$\sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n} = \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \cdots$$

is called the principal part of f at z_0. We will classify isolated singularities based on whether the principal part is zero, nonzero w/ finitely many terms, or nonzero w/ infinitely many terms. The goal is to understand how to compute residues based on the type of singularity

Definition (Types of Singularities) Suppose that f has an isolated singularity at $z_0 \in \mathbb{C}$.

(1) z_0 is an removable singularity if the principal part of f at z_0 is zero, that is, $b_n = 0 \;\; \forall n \geq 1$.

(2) z_0 is an essential singularity if the principal part

part of f at z_0 has infinitely many nonzero terms.

(3) z_0 is a pole of order m if there exists $m \geq 1$ such that $b_m \neq 0$ and $b_n = 0$ for all $n > m$. In this case, the principal part is of the form

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}.$$

A pole of order $m=1$ is called a simple pole. //

Remark (Removable Singularities) Suppose $z_0 \in \mathbb{C}$ is a removable singularity. By definition, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + 0$$

on some annulus $0 < |z-z_0| < R$. If we define

$$g(z) = \begin{cases} f(z), & z \neq z_0 \\ a_0, & z = z_0 \end{cases}$$

then $g(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ on the disk $|z-z_0| < R$.

Hence, $g(z)$ is analytic on the disk and agrees w/ f everywhere on the annulus $0 < |z-z_0| < R$. So, the singularity has been removed. //

Example

(1) (c.f. Pset 6 P4) You proved that

$$g(z) = \begin{cases} \frac{1-\cos z}{z^2}, & z \neq 0 \\ \frac{1}{2}, & z = 0 \end{cases}$$

is entire. Consider $f(z) = \frac{1-\cos z}{z^2}$. Then f has an isolated singularity at $z_0 = 0$. We can find a Laurent series on the annulus $0<|z|<\infty$. We have

$$\frac{1}{z^2}(1-\cos z) = \frac{1}{z^2}\left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\right)$$
$$= -\frac{1}{z^2}\sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = -\sum_{n=1}^{\infty} \frac{(-1)^n z^{2n-2}}{(2n)!}$$
$$= \frac{1}{2} - \frac{1}{4!}z^2 + \cdots$$

So the principal part of f at 0 is 0. Thus, f has a removable singularity.

(2) The function $f(z) = \dfrac{1-\cosh z}{z^2}$ also has a removable singularity at $z_0 = 0$. The Laurent series on $0<|z|<\infty$ is

$$\frac{1}{z^2}(1-\cosh z) = \frac{1}{z^2}\left(1 - \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}\right)$$
$$= -\sum_{n=1}^{\infty} \frac{z^{2n-2}}{(2n)!}$$

So the principal part of f at 0 is zero and so f has a removable singularity.

(3) The function $f(z) = e^{\frac{1}{z}}$ has an isolated singularity at $z_0 = 0$. The Laurent series of f on $0<|z|<\infty$ is

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{z^n \cdot n!} = 1 + \frac{1}{z\cdot 1!} + \frac{1}{z^2 \cdot 2!} + \cdots +$$

So f has an essential singularity at $z_0 = 0$.

(4) The function $f(z) = \dfrac{1}{z^2(1-z)}$ has an isolated singularity at $z_0 = 0$. The Laurent series of f on

the annulus $0<|z|<1$ is given by

$$\frac{1}{z^2}\cdot\frac{1}{1-z} = \frac{1}{z^2}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-2}$$

$$= \underbrace{\frac{1}{z^2}+\frac{1}{z}}_{\text{principal part of } f} + \sum_{n=0}^{\infty} z^n$$

Evidently, $z_0=0$ is a pole of order $m=2$.

(5) The function $f(z)=\dfrac{z^2+z-2}{z+1}$ has an isolated singularity at $z_0=-1$. The Laurent series on the annulus $0<|z+1|<\infty$ is given by

$$\frac{z^2+z-2}{z+1} = \frac{z(z+1)-2}{z+1} = z-\frac{2}{z+1}$$

$$= -1 + z+1 - \frac{2}{z+1}.$$

Hence, $z_0=-1$ is a simple pole (a pole of order $m=1$). //

Residues at Poles

The next theorem gives a characterization of poles and an efficient method for computing the residue at a pole.

Theorem (Residue at a Pole) Let $z_0\in\mathbb{C}$ be an isolated singularity of f. Then the following are equivalent:

(a) z_0 is a pole of order m.

(b) $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ for some function ϕ that is analytic and nonzero at z_0.

Moreover, when (a) and (b) are true, the residue of f at z_0 is given by

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}}$$

Proof. ((a) $\Rightarrow$ (b)) Assume z_0 is a pole of order $m \geq 1$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

with $b_m \neq 0$ on an annulus $0 < |z-z_0| < R$. Define

$$\phi(z) = \begin{cases} (z-z_0)^m f(z), & z \neq z_0 \\ b_m, & z = z_0 \end{cases}.$$

It is clear that $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$. Moreover, $\phi(z)$ has a power series on the disk $|z-z_0| < R$:

$$\phi(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} + b_1 (z-z_0)^{m-1} + b_2 (z-z_0)^{m-2} + \cdots + b_m$$

Hence, ϕ is analytic on the disk and hence at z_0. Moreover, $\phi(z_0) = b_m \neq 0$. So this proves (b).

((b) $\Rightarrow$ (a)) Assume $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ for some function ϕ that is analytic and nonzero at z_0. Since ϕ is analytic at z_0, there is a disk $|z-z_0| < R$ on which ϕ has a

Taylor series. Hence,

$$f(z) = \frac{1}{(z-z_0)^m} \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!} (z-z_0)^n$$

$$= \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!} (z-z_0)^{n-m}$$

$$= \frac{\phi(z_0)}{(z-z_0)^m} + \frac{\phi^{(1)}(z_0)}{1!\,(z-z_0)^{m-1}} + \cdots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!\,(z-z_0)} + \sum_{n=0}^{\infty} \frac{\phi^{(n+m)}(z_0)}{(n+m)!}(z-z_0)^n.$$

Since $\phi(z_0) \neq 0$ by assumption, this proves that f has a pole of order m at z_0. Moreover, it is clear that

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

■

Example

(1) The function $f(z) = \frac{z+4}{z^2+1}$ has isolated singularities at $z = i, -i$.

First, consider $z=i$. Define $\phi(z) = \frac{z+4}{z+i}$. Then clearly $f(z) = \frac{\phi(z)}{z-i}$. Moreover ϕ is analytic and nonzero at $z=i$. By the theorem, $z=i$ is a simple pole. The residue is given by

$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = \frac{i+4}{2i}$$

When $z=-i$, take $\phi(z) = \frac{z+4}{z-i}$. Then $f(z) = \frac{\phi(z)}{z+i}$, and ϕ is nonzero and analytic at $z=-i$. Hence,

$z=-i$ is a simple pole and

$$\underset{z=-i}{\text{Res}} = \phi(-i) = \frac{-i+4}{-2i}.$$

(2) The function $f(z) = \dfrac{z^3+2z}{(z-i)^3}$ has an isolated singularity at $z=i$. Consider $\phi(z) = z^3+2z$. Then $f(z) = \dfrac{\phi(z)}{(z-i)^3}$. Moreover, ϕ is analytic and nonzero at $z=i$. Hence, $z=i$ is a pole of order $m=3$.

The residue is given by

$$\underset{z=i}{\text{Res}}\, f(z) = \frac{\phi^{(3-1)}(i)}{(3-1)!} = \frac{1}{2}(6z)\Big|_{z=i}$$

$$= 3i.$$

(3) The function $f(z) = \dfrac{(\log z)^3}{z^2+1}$ has isolated singularities at $z = i, -i$. Here $\log z$ is the branch

$$\log z = \ln r + i\theta \quad (r>0 \;,\; 0<\theta<2\pi).$$

Consider $z=i$. Define $\phi(z) = \dfrac{(\log z)^3}{z+i}$. Then

$$f(z) = \frac{\phi(z)}{z-i}.$$

Clearly ϕ is analytic at $z=i$ and $\phi(i) \neq 0$ since

$$\phi(i) = \frac{(\log i)^3}{2i} = \frac{(\ln|i| + i\pi/2)^3}{2i}$$

$$= \frac{i^3 \cdot \frac{\pi^3}{8}}{2i} = -\frac{\pi^3}{16} \neq 0.$$

By the theorem, $z=i$ is a simple pole. The residue is given by

$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = \frac{-\pi^3}{16}.$$

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Zeros of Analytic Functions

Definition (Zeros of Analytic Functions) Assume f is analytic at $z_0 \in \mathbb{C}$. We say that f has a *zero of order m* if $f(z_0)=0$ and there exist $m \geq 1$ such that $f^{(m)}(z_0) \neq 0$ but $f^{(n)}(z_0) = 0$ for all $0 \leq n < m$. A zero is isolated if there exists $\varepsilon > 0$ such that $f(z) \neq 0$ for all $z \in D_\varepsilon(z_0) \setminus \{z_0\}$.

$\boxed{\text{Theorem (Characterization of Zeros)}}$ Suppose that f is analytic at z_0. The following are equivalent:

(a) z_0 is a zero of f of order m.

(b) $f(z) = (z-z_0)^m g(z)$ for some function $g(z)$ analytic and nonzero at z_0.

Proof. ($a \Rightarrow b$) Assume z_0 is a zero of order m. Since f is analytic at z_0, f has a Taylor series on some disk $D_\varepsilon(z_0)$:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

$$= \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

$$= (z-z_0)^m \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-m}.$$

Define $g(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$. Clearly $g(z)$ is analytic at z_0 since it converges on $D_\varepsilon(z_0)$. Moreover, $g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0$ since z_0 is a pole of order m.

($b \Rightarrow a$) Assume $f(z) = (z-z_0)^m g(z)$ where g is analytic and nonzero at z_0. Since g is analytic at z_0, there is a disk $D_\varepsilon(z_0)$ on which it has a Taylor series. Then

$$f(z) = (z-z_0)^m g(z)$$

$$= (z-z_0)^m \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^n$$

$$= \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^{n+m}.$$

Since Taylor series are unique, the coefficients in this power series for f are the ones given by Taylors theorem. Hence,

$$\frac{f^{(n)}(z_0)}{n!} = 0 \quad \text{for all} \quad n = 0, 1, \dots, m-1$$

and $\frac{f^{(m)}(z_0)}{m!} = g(z_0)$. Hence, $f^{(m)}(z_0) = g(z_0) \cdot m! \neq 0$

and $f^{(n)}(z_0) = 0$ for all $0 \leq n < m-1$. ∎

$\boxed{\text{Example}}$ The function $p(z) = z^3 - 1$ has a zero of order $m = 1$ at $z_0 = 1$. Just define $g(z) = z^2 + z + 1$. Then

$$p(z) = (z-1)(z^2+z+1) = (z-1)g(z).$$

Clearly $g(z)$ is analytic at $z_0 = 1$ and

$$g(1) = 3 \neq 0.$$

So by the theorem $p(z)$ has a zero of order 1 at $z_0 = 1$.

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$\boxed{\text{Theorem (Zeros of nonzero Analytic Functions)}}$ Suppose that

(a) f is analytic at z_0;

(b) $f(z_0) = 0$, but f is not identically zero on any neighborhood of z_0.

Then z_0 is an isolated zero of f.

Proof. By (a) there is a disk $|z - z_0| < R$ on which we can write

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

If $f^{(n)}(z_0) = 0$ for all $n \geq 0$, then $f(z)$ would be identically zero on $|z - z_0| < R$, contrary to (b). Hence, there exists $m \geq 1$ such that

$$f(z_0) = f^{(1)}(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \quad \text{but}$$

$f^{(m)}(z_0) \neq 0$. Hence f has a zero of order m. Then

$$f(z) = (z - z_0)^m g(z)$$

for some function $g(z)$ that is analytic and nonzero at z_0. Since g is continuous and nonzero at z_0, there exists a disk $D_\varepsilon(z_0)$ on which $g(z) \neq 0$ for all $z \in D_\varepsilon(z_0)$. Hence, $f(z) \neq 0$ on the deleted disk $D_\varepsilon(z_0) \setminus \{z_0\}$. Hence, z_0 is an isolated zero of f.

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Zeros and Poles

Theorem (Zeros and Poles) Suppose that

(a) $p(z)$ and $q(z)$ are analytic at $z_0 \in \mathbb{C}$.

(b) $p(z_0) \neq 0$ and $q(z)$ has a zero of order m at z_0.

Then $f(z) = \dfrac{p(z)}{q(z)}$ has a pole of order m at z_0.

Proof. First, since $q(z)$ is analytic at z_0 and has a zero of order m at z_0, by the preceding theorem z_0 is an isolated zero. Hence f has an isolated singularity at z_0. Since z_0 is a zero of order m, choose $g(z)$ that is analytic and nonzero at z_0 such that

$$q(z) = (z - z_0)^m g(z).$$

Hence, write $\phi(z) = \dfrac{p(z)}{g(z)}$ so that we have

$$f(z) = \frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{(z-z_0)^m} = \frac{\phi(z)}{(z-z_0)^m}.$$

Moreover, $\phi(z_0) \neq 0$ and it is analytic at z_0 since both $p(z)$ and $g(z)$ are. Hence f has a pole of order m. ∎

$\boxed{\text{Example}}$ Consider $f(z) = \frac{1}{1-\cos z}$. Using the theorem, we can show that f has a pole of order $m=2$ at $z_0 = 0$.

Let $p(z) = 1$ and $q(z) = 1 - \cos z$. Clearly, both $p(z)$ and $q(z)$ are analytic at $z_0 = 0$. Moreover,

$$p(0) = 1 \neq 0$$

and $q(z)$ has a zero of order $m=2$ since

$$q(0) = 1 - \cos 0 = 0$$
$$q'(0) = \sin 0 = 0$$
$$q''(0) = \cos 0 = 1 \neq 0,$$

By the theorem, $f(z)$ has a pole of order $m=2$ at $z_0 = 0$.

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$\boxed{\text{Theorem (Residue at a Simple Pole)}}$ Suppose $p(z), q(z)$ are analytic at z_0. If

$$p(z_0) \neq 0, \quad q(z_0) = 0, \quad \text{and} \quad q'(z_0) \neq 0,$$

then z_0 is a simple pole of $\frac{p(z)}{q(z)}$ and

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

Proof. First, $q(z)$ has a zero of order $m=1$ since $q(z_0) = 0$

and $g'(z_0) \neq 0$. Choose a function $g(z)$ that is analytic and nonzero at z_0 such that

$$q(z) = (z - z_0)\, g(z). \quad (*)$$

Using $\phi(z) = \dfrac{p(z)}{g(z)}$ we can conclude that

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{z - z_0}$$

(see proof of preceding thm)

and that $\dfrac{p(z)}{q(z)}$ has a pole of order $m=1$. Hence,

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \phi(z_0) = \frac{p(z_0)}{g(z_0)}.$$

But from $(*)$, we obtain

$$q'(z_0) = q'(z)\Big|_{z=z_0} = \frac{d}{dz}(z-z_0)\,g(z)\Big|_{z=z_0}$$

$$= \left[g(z) + g'(z)(z-z_0) \right]\Big|_{z=z_0}$$

$$= g(z_0).$$

Hence,

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}. \quad \blacksquare$$

$\boxed{\text{Example}}$

(1) Consider $f(z) = \cot z = \dfrac{\cos z}{\sin z}$. Let $p(z) = \cos z$ and $q(z) = \sin z$. Let $z_k = k\pi$, $k \in \mathbb{Z}$. Clearly, both p and q are analytic at z_k, because they are entire. Moreover,

$$p(z_k) = \cos k\pi = (-1)^k \neq 0$$

$$q(z_k) = \sin k\pi = 0$$

$$q'(z_k) = \cos k\pi = (-1)^k \neq 0.$$

Hence, z_K is a simple pole for each $K \in \mathbb{Z}$ and

$$\operatorname*{Res}_{z=z_K} \cot z = \frac{p(z_K)}{q'(z_K)} = \frac{(-1)^K}{(-1)^K} = 1.$$

Let C be the positively oriented circle of radius $K\pi + 1$ centered at 0. Then

$$\int_C \cot z \, dz = 2\pi i \sum_{n=-K}^{K} \operatorname*{Res}_{z=z_K} \cot z$$
$$= 2\pi i (2K+1).$$

(2) Consider $f(z) = \dfrac{z - \sinh z}{z^2 \sinh z}$. Consider $z = \pi i$ and

let $p(z) = z - \sinh z$ and $q(z) = z^2 \sinh z$. Both p and q are entire and hence analytic at $z = \pi i$. Moreover,

$$p(\pi i) = \pi i - \sinh \pi i = \pi i \neq 0$$
$$q(\pi i) = (\pi i)^2 \sinh \pi i = 0$$
$$q'(\pi i) = \left(2z \sinh z + z^2 \cosh z\right)\Big|_{z=\pi i}$$
$$= (\pi i)^2 \cosh \pi i$$
$$= -\pi^2 \left(\frac{e^{\pi i} + e^{-\pi i}}{2}\right)$$
$$= -\pi^2 \left(\frac{-1 - 1}{2}\right) = \pi^2 \neq 0.$$

So $z = \pi i$ is a simple pole and

$$\operatorname*{Res}_{z=\pi i} f(z) = \frac{p(\pi i)}{q'(\pi i)} = \frac{\pi i}{\pi^2} = \frac{i}{\pi}.$$

(3) Consider $f(z) = \dfrac{z}{z^4 + 4}$. Consider $z = 1 + i$. Let

$p(z) = z$ and $q(z) = z^4 + 4$. then

$$p(1+i) = 1+i \neq 0$$
$$q(1+i) = 0$$
$$q'(1+i) = 4(1+i)^3 \neq 0.$$

So $z = 1+i$ is a simple pole and

$$\operatorname*{Res}_{z=1+i} f(z) = \frac{p(1+i)}{q'(1+i)} = \frac{1}{4(1+i)^2}.$$

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